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7. Exercise on Convex Optimization

Problem 24: (Linear scaling invariance of Newton's method)

Show that Newton's method is unaffected by linear scaling of the variables. Consider a linear invertible transformation of variables $x = Sy$. Write Newton's method in the space of the Variables y and show that it generates the sequence $y^k = S^{-1}x^k$, where x^k is the sequence generated by Newton's method in the space of the variables x.

Problem 25: (Newton's method and Armijo rule)

a) Consider the pure form of Newton's method for the case of the cost function $f(x) := \|x\|^\beta$, where $\beta > 1$. For what starting points and values of β does the method converge to the optimal solution? What happens when $\beta \leq 1$?

b) Repeat the previous part for the case where Newton's method with the Armijo rule is used.

Problem 26: (Least square problem)

Consider the least square problem

$$\begin{aligned} \text{minimize} \quad & f(x) = \frac{1}{2}\|g(x)\|^2 = \frac{1}{2}\sum_{i=1}^{m}\|g_i(x)\|^2 \\ \text{subject to} \quad & x \in \mathbb{R}^n, \end{aligned} \tag{0.2}$$

where g is a continuously differentiable function with component functions $g_1, \dots, g_m : \mathbb{R}^n \to \mathbb{R}$ with $m < n$.

a) Show that the Hessian matrix is singular at any optimal solution x^* for which $g(x^*) = 0$.

b) Consider the case where g is linear and of the form $g(x) = z - Ax$, where A is an $m \times n$ matrix. Show that there are infinitely many optimal solutions. Show also that if A has linearly independent rows, $x^* = A'(AA')^{-1}z$ is one of these solutions.

Problem 27: (Computational problem)

Consider the three-dimensional problem

$$\begin{aligned} \text{minimize} \quad & f(x) = \frac{1}{2}\left(x_1^2 + x_2^2 + 0.1x_3^2\right) + 0.55x_3 \\ \text{subject to} \quad & 1 = x_1 + x_2 + x_3, \quad 0 \leq x_1,\ 0 \leq x_2,\ 0 \leq x_3. \end{aligned} \tag{0.3}$$

Show that the global minimum is $x^* = (1/2, 1/2, 0)$. Write a computer program implementing the conditional gradient method with the line minimization step size rule. (Here, there is a closed form expression for the minimizing step size) Verify computationally that for a starting point (ξ_1, ξ_2, ξ_3) with $\xi_i > 0$ for all i, and $\xi_1 \neq \xi_2$, the rate of convergence is not linear in the sense that

$$\lim_{k\to\infty} \frac{f(x^{k+1}) - f(x^*)}{f(x^k) - f(x^*)} = 1.$$